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TROPICAL MATRIX DUALITY AND GREEN'S $\mathscr{D}$ RELATION

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Abstract. We give a complete description of Green's $\mathscr{D}$ relation for the multiplicative semigroup of all $n \times n$ tropical matrices. Our main tool is a new variant on the *duality* between the row and column space of a tropical matrix (studied by Cohen, Gaubert and Quadrat and separately by Develin and Sturmfels). Unlike the existing duality theorems, our version admits a converse, and hence gives a necessary and sufficient condition for two tropical convex sets to be the row and column space of a matrix. We also show that the matrix duality map induces an isometry (with respect to the Hilbert projective metric) between the projective row space and projective column space of any tropical matrix, and establish some foundational results about Green's other relations.

Tropical algebra (also known as max-plus algebra or max-algebra) is the algebra of the real numbers (sometimes augmented with $-\infty$ and/or $+\infty$) when equipped with the binary operations of addition and maximum. It has traditional applications in a wide range of subjects, such as combinatorial optimisation and scheduling problems [6], analysis of discrete event systems [20], control theory [9], formal language and automata theory [27, 29] and combinatorial/geometric group theory [3]. More recently, exciting connections have emerged with algebraic geometry [2, 25, 28]; these have also led to new applications in areas such as phylogenetics [15] and statistical inference [26]. The first detailed axiomatic study was conducted by Cuninghame-Green [12] and this theory has been developed further by a number of researchers (see [1, 21] for surveys).

Since many of the problems which arise in application areas are naturally expressed in terms of (max-plus) linear equations, much of tropical algebra is concerned with matrices. Many researchers have had cause to prove *ad hoc* results about the *multiplication* of tropical matrices; there has also been considerable attention paid to certain special questions such as Burnside-type problems for semigroups of tropical matrices [13, 17, 27, 29]. Surprisingly, though, there has been relatively little systematic study of these semigroups, and little is known about their abstract algebraic structure. In particular, there has been until recently no understanding of the semigroup of all matrices of a given size over the tropical semiring, comparable with the classical theory of the general linear group or full matrix semigroup over a field. The detailed study of this

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semigroup was recently initiated by Johnson and the second author [23] and independently by Izhakian and Margolis [22].

Green's relations [8,19] are five equivalence relations ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$) and three pre-orders ($\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$ and $\leq_{\mathscr{J}}$) which can be defined upon any semigroup, and which encapsulate the structure of its principal left, right and two-sided ideals and maximal subgroups. They are amongst the most powerful tools for understanding the structure of semigroups and monoids, and play a key role in almost every aspect of modern semigroup theory. The relations $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$, $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ are easily described for matrix semigroups over arbitrary semirings with identity and zero. For example, two matrices are $\mathscr{R}$-related exactly if they have the same column space; see [23] or Section 3 below for a more detailed explanation.

The relations $\mathscr{D}$, $\mathscr{J}$ and $\leq_{\mathscr{J}}$, however, are rather more subtle. In the classical case of a finite dimensional full matrix semigroup over a field, it is well known that $\mathscr{D}$ and $\mathscr{J}$ coincide and encapsulate the concept of *rank*, with the pre-order $\leq_{\mathscr{J}}$ corresponding to the obvious order on ranks. Johnson and the second author [23] showed that $\mathscr{D}$ and $\mathscr{J}$ also coincide for the 2×2 tropical matrix semigroup, with the $\mathscr{J}$-class of a matrix being determined by the *isometry type* of its row space (or equivalently, its column space).

The main aim of the present paper is to give a complete description of Green's $\mathscr{D}$ relation for the full matrix semigroup of arbitrary finite dimension over the tropical semiring. Specifically, we show that two matrices are $\mathscr{D}$-related exactly if their row spaces (or equivalently, their column spaces) are isomorphic as semimodules (Theorems 5.1 and 5.5 below).

While our main result develop those from the 2×2 case, proved in [23], the methods used here are entirely different. The results of [23] were obtained naively by direct algebraic manipulations, which are unenlightening even in two dimensions, and quickly become impractical in higher dimensions. Here, our approach is geometric, with the main tool being the phenomenon of *duality* between the row space and column space of a tropical matrix. It is well known that there is a natural and canonical bijection between the row space and column space of a given tropical matrix which preserves certain aspects of their structure. Specifically, Cohen, Gaubert and Quadrat [10] have showed that it is an antitone lattice isomorphism, while Develin and Sturmfels [14] have proved that it induces a combinatorial isomorphism of (Euclidean) polyhedral complexes. Here we shall employ a slight variation of the duality theorem, which states that the duality map is in a certain algebraic sense an *anti-isomorphism* of semimodules (Theorem 2.4). The important thing for our purpose is that this version of duality admits a converse: any two finitely generated convex sets which are anti-isomorphic must be the row and column space of a tropical matrix (Theorem 4.4). As a corollary of our duality theorem, we also observe that the duality map is an isometry (with respect to the Hilbert projective metric) between the row and column spaces of any tropical matrix. We believe that these results are likely to be of independent interest.

In addition to this introduction, the paper comprises six sections. Section 1 provides a brief summary of the necessary background material from tropical

algebra and geometry. Section 2 introduces and proves our new version of the tropical duality theorem, and discusses briefly its relationship to existing results. Section 3 recalls the definitions of Green's relations, and establishes some basic properties of Green's relations in finite dimensional full matrix semigroups over the tropical semirings. Section 4 proves our converse to the duality theorem. Section 5 applies in the preceding results to describe Green's $\mathscr{D}$ relation in these semigroups. Finally, Section 6 contains some remarks on questions remaining open and the potential application of our methods in wider contexts.

Because of a recent proliferation of applications, tropical mathematics is now of interest to a broad range of researchers with radically different motivations and backgrounds. This article is likely to be of interest to many of these people and, in addition, to abstract semigroup theorists with no experience of tropical mathematics. For this reason, we have endeavoured to keep the article self-contained by minimising the use of specialist terminology, notation and machinery, and by including elementary proofs for foundational results where feasible. In so doing we crave the indulgence of specialists in this area, who may feel that parts of the paper could be expressed more concisely.

1. Preliminaries

The *finitary tropical semiring* $\mathbb{FT}$ is the semiring (without additive identity) consisting of the real numbers under the operations of addition and maximum. The *tropical semiring* $\mathbb{T}$ is the finitary tropical semiring augmented with an extra element $-\infty$, which acts as a zero for addition and an identity for maximum. The *completed tropical semiring* $\overline{\mathbb{T}}$ is the tropical semiring augmented with an extra element ∞, which acts as a zero for both maximum and addition, save that $(-\infty) + \infty = \infty + (-\infty) = -\infty$. These structures share many of the properties of fields, with addition and maximum playing the roles of field multiplication and field addition respectively; for this reason we write $a \oplus b$ for $\max(a, b)$ and $a \otimes b$ or just ab for $a + b$. In particular, note that $\otimes$ distributes over $\oplus$ and that both operations are commutative and associative. Hence, they give rise to a natural, associative multiplication on matrices over $\overline{\mathbb{T}}$.

We extend the usual order $\leq$ on the reals to a total order on $\overline{\mathbb{T}}$ in the obvious way, with $-\infty < x < \infty$ for all $x \in \mathbb{R}$, noting that $a \oplus b = a$ exactly if $b \leq a$ and that $\overline{\mathbb{T}}$ under $\leq$ is a complete lattice. The semirings $\mathbb{FT}$ and $\overline{\mathbb{T}}$ admit a natural order-reversing involution, $x \mapsto -x$. In $\mathbb{FT}$ this involution obviously distributes over $\otimes$ (so $-(xy) = (-x)(-y)$), but more caution is required in $\overline{\mathbb{T}}$ where $-(\infty(-\infty)) \neq (-\infty)\infty$. The involution has the following elementary property, which is obvious in $\mathbb{FT}$ but needs to be verified by case analysis in $\overline{\mathbb{T}}$.

Proposition 1.1. *For any $a, b, c \in \overline{\mathbb{T}}$ we have*

$$ab \leq c \iff a(-c) \leq (-b).$$

Proof. We suppose that $ab \leq c$ and show that $a(-c) \leq (-b)$, the reverse implication being dual. If a, b and c are all finite then the claim is immediate. If $a = -\infty$ then $a(-c) = -\infty$ so the desired inequality holds. If $a = \infty$ then for $ab \leq c$ we must have either $b = -\infty$ or $c = \infty$; in both cases the desired inequality again holds. If $b = -\infty$ then the claim is immediate, while if $b = \infty$

then we must have either $a = -\infty$ or $c = \infty$, again guaranteeing the claim. Finally, if $c = \infty$ then the claim is immediate, while if $c = -\infty$ then we must have $a = -\infty$ or $b = -\infty$, placing us in a case we have already considered. $\square$

For $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$ we are interested in the space S^n of *(affine tropical) vectors*. We write x_i for the ith component of a vector $x \in S^n$. We extend $\oplus$ and $\leq$ to S^n componentwise, so that $(x \oplus y)_i = x_i \oplus y_i$, and $x \leq y$ exactly if $x_i \leq y_i$ for all i. Sometimes we wish to stress that a space S^n is composed of row vectors or of column vectors, in which case we write $S^{1\times n}$ or $S^{n\times 1}$ respectively. We define a scaling action of S on S^n by

$$\lambda(x_1, \ldots, x_n) = (x_1 + \lambda, \ldots, x_n + \lambda)$$

for each $\lambda, x_1, \ldots, x_n \in S$. From affine tropical n-space we obtain *projective tropical* $(n-1)$*-space* (denoted $\mathbb{PFT}^{n-1}$, $\mathbb{PT}^{n-1}$ or $\overline{\mathbb{PT}}^{n-1}$ as appropriate) by identifying two vectors if one is a tropical multiple of the other by an element of $\mathbb{FT}$.

An (S-linear) *convex set* in S^n is a subset closed under $\oplus$ and scaling by elements of S, that is, a linear subspace of S^n. If $X \subseteq S^n$ then the (S-linear) *span* or *convex hull* of T is the set of all vectors which can be written as tropical linear combinations (with scaling and vector addition $\oplus$ as defined above) of finitely many vectors from X with coefficients drawn from S. It is easily seen that if $X \subseteq \mathbb{FT}^n$ then the $\mathbb{FT}$-linear span of X is the intersection with $\mathbb{FT}^n$ of the $\mathbb{T}$-linear span of X. Similarly, if $X \subseteq \mathbb{T}^n$ then the $\mathbb{T}$-linear span of X is the intersection with $\mathbb{T}^n$ of the $\overline{\mathbb{T}}$-linear span of X. Each convex set X induces a subset of the corresponding projective space, termed the *projectivisation* of X. Convex sets in tropical space have a very interesting structure, which has been extensively studied (see for example [4, 7, 10, 11, 14, 16, 18, 24]) but is still not fully understood.

We define a scalar product operation $\overline{\mathbb{T}}^n \times \overline{\mathbb{T}}^n \to \overline{\mathbb{T}}$ on completed tropical n-space $\overline{\mathbb{T}}^n$ by setting

$$\langle a \mid b \rangle = \max\{\lambda \in \overline{\mathbb{T}} \mid \lambda a \leq b\}.$$

The existence of such a maximum is easily verified by case analysis. This operation is a *residual* operation in the sense of residuation theory [5]; it has been quite extensively used in tropical mathematics (see for example [10]). We shall need some elementary properties of this operation.

Proposition 1.2. *For any $n \geq 1$ and $x, y \in \overline{\mathbb{T}}^n$,*

$$\langle x \mid y \rangle = -\left(\bigoplus_{i=1}^{n} \{x_i \otimes (-y_i)\} \right).$$

Proof. First, notice that if $x_i = -\infty$ or $y_i = \infty$ then $x_i(-y_i) = -\infty$ and $\lambda x_i \leq y_i$ for all λ, so the ith component makes no contribution to the maximum in the statement and no difference to the maximum in the definition of $\langle x \mid y \rangle$. If *all* components are like this, then it is readily verified that both $\langle x \mid y \rangle$ and the right hand side in the statement take the value ∞, so the proposition holds. Otherwise, we may discard any such components and seek to prove the claim only in the case that $x_i \neq -\infty$ and $y_i \neq \infty$ for all i.

If $x_i = \infty$ [respectively $y_i = -\infty$] then since we are assuming $y_i \neq \infty$ [$x_i \neq -\infty$] we have $x_i(-y_i) = \infty$ so that

$$-\bigoplus_{i=1}^{n}\{x_i(-y_i)\} = -\infty.$$

We also have $\lambda x_i > y_i$ for all $\lambda > -\infty$, so that $\langle x \mid y\rangle = -\infty$.

Thus, we may assume that $x, y \in \mathbb{FT}^n$. Now by definition we have $\lambda x \leq y$ if and only if $\lambda x_i \leq y_i$ for all i. By Proposition 1.1 this is true exactly if $\lambda(-y_i) \leq (-x_i)$, that is, if and only if $-\lambda \geq x_i - y_i$ for all i. It follows that $\max\{x_i - y_i\}$ is the smallest $-\lambda$ such that $\lambda x \leq y$, and so $-\max\{x_i - y_i\}$ is the largest λ such that $\lambda x \leq y$, which is by definition $\langle x \mid y\rangle$. □

Proposition 1.3. *Let* $x, y \in \overline{\mathbb{T}}^n$. *Then* $\langle x \mid y\rangle = \langle -y \mid -x\rangle$

Proof. By Proposition 1.1 we have $\lambda(-y_i) \leq (-x_i)$ for all i if and only if $\lambda x_i \leq y_i$ for all i. Thus, $\lambda(-y) \leq (-x)$ if and only if $\lambda x \leq y$, so

$$\langle x \mid y\rangle = \max\{\lambda \mid \lambda x \leq y\} = \max\{\lambda \mid \lambda(-y) \leq (-x)\} = \langle -y \mid -x\rangle.$$

□

Proposition 1.4. *For any* $x, y \in \overline{\mathbb{T}}^n$, *we have* $x \leq y$ *if and only if* $\langle x \mid y\rangle \geq 0$.

Proof. If $x \leq y$ then $0x \leq y$ so by the definition of $\langle x \mid y\rangle$ we have $0 \leq \langle x \mid y\rangle$. Conversely, if $\langle x \mid y\rangle \geq 0$ then $x = 0x \leq \langle x \mid y\rangle x \leq y$. □

Proposition 1.5. *Let* $n \geq 1$ *and* $a, r_1, \dots, r_k \in \overline{\mathbb{T}}^n$ *be such that* a *lies in the convex hull of* $r_1, \dots, r_k$. *Then*

$$a = \bigoplus_{i=1}^{k}\langle r_i|a\rangle r_i.$$

Proof. For $1 \leq j \leq n$, let b_j be the jth component of the right hand side; we must show that each $a_j = b_j$. Since a lies in the convex hull of $r_1, \dots, r_k$, it can be written in the form

$$a = \bigoplus_{i=1}^{k} \lambda_i r_i$$

for some scalars $\lambda_i \in \overline{\mathbb{T}}$. Then for $1 \leq j \leq n$ we have $a_j = \bigoplus_{i=1}^{k} \lambda_i (r_i)_j$, so there is an i such that $a_j = \lambda_i (r_i)_j$. Now $\lambda_i r_i \leq a$ so by the definition of $\langle r_i \mid a\rangle$ we have $\lambda_i \leq \langle r_i|a\rangle$, and so $a_j = \lambda_i (r_i)_j \leq \langle r_i|a_i\rangle (r_i)_j \leq b_j$.

Conversely, it follows from the definition of $\langle r_i \mid a\rangle$ that $\langle r_i \mid a\rangle r_i \leq a$ for all i. Thus, $\langle r_i \mid a\rangle (r_i)_j \leq a_j$ for all i and j, and hence $b_j \leq a_j$ for all j. □

We define a distance function on $\overline{\mathbb{T}}^n$ by $d_H(x, y) = 0$ if x is a finite scalar multiple of y, and

$$d_H(x, y) = -(\langle x \mid y\rangle \otimes \langle y \mid x\rangle)$$

otherwise. It is easily verified that the map d_H is invariant under scaling x or y by finite tropical scalars, and hence well-defined on $\overline{\mathbb{PT}}^{n-1}$. In fact, it is well known that d_H is an extended metric (that is, a metric which is permitted to take the value ∞) on projective space. We call it the *(tropical) Hilbert projective metric.*

Proposition 1.6. *d_H is an extended metric on $\overline{\mathbb{PT}}^{n-1}$.*

Proof. For the triangle inequality, suppose $x, y, z \in \overline{\mathbb{T}}^n$. If x and y are finite scalar multiples, or if y and z are finite scalar multiples then the triangle inequality is trivially satisfied. Otherwise, by the definition of the bracket we have $\langle x \mid y\rangle x \leq y$ and $\langle y \mid z\rangle y \leq z$. Thus,

$$(\langle x \mid y\rangle + \langle y \mid z\rangle)x = \langle y \mid z\rangle\langle x \mid y\rangle x \leq \langle y \mid z\rangle y \leq z$$

which by the definition of $\langle x \mid z\rangle$ means that $\langle x \mid y\rangle + \langle y \mid z\rangle \leq \langle x \mid z\rangle$. A symmetrical argument shows that $\langle z \mid y\rangle + \langle y \mid x\rangle \leq \langle z \mid x\rangle$, and combining these we have

$$\begin{aligned} d_H(x, y) + d_H(y, z) &= -(\langle x \mid y\rangle + \langle y \mid x\rangle) - (\langle y \mid z\rangle + \langle z \mid y\rangle) \\ &= -(\langle x \mid y\rangle + \langle y \mid z\rangle + \langle z \mid y\rangle + \langle y \mid x\rangle) \\ &\geq -(\langle x \mid z\rangle + \langle z \mid x\rangle) \\ &= d_H(x, z). \end{aligned}$$

The other conditions are readily verified from the definition. □

Notice that any map θ from a subspace of $\overline{\mathbb{T}}^p$ to a subspace of $\overline{\mathbb{T}}^q$ which either preserves the bracket ($\langle x \mid y\rangle = \langle \theta(x) \mid \theta(y)\rangle$) or reverses the bracket ($\langle x \mid y\rangle = \langle \theta(y) \mid \theta(x)\rangle$) will preserve the Hilbert projective metric. We call such maps *orientation-preserving* and *orientation-reversing*, respectively.

2. Duality

In this section we begin by providing a brief introduction to tropical matrix duality, We introduce the notion of an *anti-isomorphism* of semimodules, establish some basic properties of anti-isomorphisms, and show that the matrix duality map is an anti-isomorphism.

Let $S = \mathbb{FT}$, $S = \mathbb{T}$ or $S = \overline{\mathbb{T}}$. Let $p, q \geq 1$ and A be a $p \times q$ matrix over S, with rows $A_1, \dots, A_p \in S^{1\times q}$ and columns $B_1, \dots, B_q \in S^{p\times 1}$. We define the ($S$-linear) *row space* $R_S(A)$ of A to be the S-linear convex hull of the vectors $A_1, \dots, A_p$. Dually, the *(S-linear) column space* $C_S(A)$ is the S-linear convex hull of the vectors $B_1, \dots, B_q$. Since we are most often interested in the case $S = \overline{\mathbb{T}}$, we shall for brevity write $C(A)$ and $R(A)$ for $C_{\overline{\mathbb{T}}}(A)$ and $R_{\overline{\mathbb{T}}}(A)$ respectively.

Now treating A as a matrix over $\overline{\mathbb{T}}$, we define a map $\theta_A : R(A) \to C(A)$ by

$$\theta_A(x) = A(-x)^T = \bigoplus_{i=1}^{q} (-x_i)B_i$$

where the second equality is immediate from the definition of matrix multiplication. Notice that, again just using the definition of matrix multiplication, the ith component of $\theta_A(x)$ is

$$\bigoplus_{j=1}^{q} \{A_{ij} + (-x_j)\},$$

which by Proposition 1.2 is exactly $-\langle A_i \mid x\rangle$.

Similarly, we define $\theta'_A : C(A) \to R(A)$ by

$$\theta'_A(y) = (-y)^T A = \bigoplus_{j=1}^{p} (-y_j) A_j.$$

There is an obvious duality between θ'_A and θ_A via the transpose map; indeed for every $y \in C(A)$ we have $y^T \in R(A^T)$ and $\theta'_A(y) = (\theta_{A^T}(y^T))^T$. From this, or directly, we may deduce that the jth component of $\theta'_A(y)$ is $-\langle B_j \mid y \rangle$.

The map θ_A, which we shall call the *duality map* of A, was studied (in a rather more general axiomatic setting, of which the completed tropical semiring $\overline{\mathbb{T}}$ is a special case) by Cohen, Gaubert and Quadrat [10], who established that it is an antitone isomorphism of lattices. Its restriction to $\mathbb{FT}$ has also been considered by Develin and Sturmfels [14] who observed that it preserves the Euclidean polytope structure of the row space. We shall need a slight strengthening of these results; we make no claim of originality in respect of this, since the stronger form can be deduced from [10] and is probably essentially known to experts in the field. However, since [10] is not very accessible to non-specialists, and we believe this paper will have a rather broader readership, we include a direct, elementary combinatorial proof.

We begin with the following elementary property of the duality map.

Proposition 2.1. *For any matrix A over $\overline{\mathbb{T}}$, the maps θ_A and θ'_A are mutually inverse bijections between $R(A)$ and $C(A)$. If the entries of A are all finite then θ_A and θ'_A restrict to mutually inverse bijections between $R_{\mathbb{FT}}(A)$ and $C_{\mathbb{FT}}(A)$.*

Proof. Suppose A is a $p \times q$ matrix, and let $B_1, \dots, B_q$ be the columns of A. We claim that $\theta_A \circ \theta'_A$ is the identity function on $C(A)$. Indeed, given $c \in C(A)$, by the observations above we have

$$\theta'_A(c) = (-\langle B_1 \mid c \rangle, \dots, -\langle B_q \mid c \rangle).$$

Now using the definition of θ_A we have

$$\theta_A(\theta'_A(c)) = \bigoplus_i (-(-\langle B_i \mid c \rangle)) B_i = \bigoplus_i \langle B_i \mid c \rangle B_i = c,$$

where the last equality is guaranteed by Proposition 1.5 because c lies in the convex hull of $B_1, \dots, B_q$. A dual argument shows that $\theta'_A \circ \theta_A$ is the identity function on $R(a)$, which completes the proof that θ_A and θ'_A are mutually inverse bijections between $R(A)$ and $C(A)$.

Now suppose all entries of A are finite. In this case, it is immediate from the definition that θ_A maps finite vectors to finite vectors, so it maps $R_{\mathbb{FT}}(A)$ into $C_{\mathbb{FT}}(A) \cap \mathbb{FT}^p$. But by our observations in Section 1 we have $C_{\mathbb{FT}}(A) = C(A) \cap \mathbb{FT}^p$, so θ_A maps $R_{\mathbb{FT}}(A)$ into $C_{\mathbb{FT}}(A)$. A dual argument shows that θ'_A maps $C_{\mathbb{FT}}(A)$ into $R_{\mathbb{FT}}(A)$. Since θ_A and θ'_A are mutually inverse bijections, it follows that they restrict to mutually inverse bijections between $C_{\mathbb{FT}}(A)$ and $R_{\mathbb{FT}}(A)$. $\square$

Notice that the duality maps θ_A and θ'_A are defined for matrices over $\mathbb{FT}$ or $\overline{\mathbb{T}}$, but do not make sense for matrices over $\mathbb{T}$ because they depend upon the involution $x \mapsto -x$. Indeed, in the special case that A is the 1×1 matrix

with single entry 0, the duality map is exactly this involution. The core of the proof of our duality theorem is the following elementary property of the bracket operation.

Lemma 2.2. *Suppose $n \geq 1$ and $a, b, r_1, \ldots r_k \in \overline{\mathbb{T}}^n$. Then*

$$\langle a \mid b \rangle \leq \langle (\langle r_1 \mid a \rangle, \ldots, \langle r_k \mid a \rangle) \mid (\langle r_1 \mid b \rangle, \ldots, \langle r_k \mid b \rangle) \rangle$$

with equality provided a and b are contained in the $\overline{\mathbb{T}}$-linear convex hull of $r_1, \ldots, r_k$.

Proof. Let $x = (\langle r_1 \mid a \rangle, \ldots, \langle r_k \mid a \rangle)$ and $y = (\langle r_1 \mid b \rangle, \ldots, \langle r_k \mid b \rangle)$. We first show that $\langle a \mid b \rangle \leq \langle x \mid y \rangle$. Firstly, if $\langle x \mid y \rangle = \infty$ then there is nothing to prove. Next, suppose $\langle x \mid y \rangle = -\infty$. By Proposition 1.2 there is an i such that $x_i(-y_i) = \infty$, which means either

(1) $x_i = \infty$ and $y_i \neq \infty$; or
(2) $x_i \neq -\infty$ and $y_i = -\infty$.

In case (1), since $\langle r_i \mid b \rangle = y_i \neq \infty$, there exists j such that $(r_i)_j \neq -\infty$ and $b_j \neq \infty$. But since $\langle r_i \mid a \rangle = x_i = \infty$ we must have $a_j = \infty$. But now $a_j(-b_j) = \infty$ which by Proposition 1.2 again ensures that $\langle a \mid b \rangle = -\infty$.

In case (2) we have $\langle r_i \mid a \rangle = x_i \neq -\infty$ and $\langle r_i \mid b \rangle = y_i = -\infty$. Applying the same argument as above to the latter, there is a j such that either

(2A) $(r_i)_j = \infty$ and $b_j \neq \infty$; or
(2B) $(r_i)_j \neq -\infty$ and $b_j = -\infty$.

In case (2A), since $(r_i)_j = \infty$ but $\langle r_i \mid a \rangle \neq -\infty$ we must have $a_j = \infty$, whereupon $a_j(-b_j) = \infty$. In case (2B), by a similar argument, we must have $a_j \neq -\infty$ and again $a_j(-b_j) = \infty$, which by Proposition 1.2 ensures that $\langle a \mid b \rangle = -\infty$.

Now consider the case in which $\langle x \mid y \rangle$ is finite. By Proposition 1.2 there is an i such that $\langle x \mid y \rangle = -(x_i(-y_i))$. Since $\langle x \mid y \rangle$ is finite, x_i and y_i must be finite, so we have $\langle x \mid y \rangle = y_i - x_i = \langle r_i \mid b \rangle - \langle r_i \mid a \rangle$. By the same argument, since $\langle r_i \mid b \rangle = y_i$ is finite, there is a j such that $\langle r_i \mid b \rangle = b_j - (r_i)_j$ with b_j and $(r_i)_j$ finite, whereupon

$$b_j = \langle r_i \mid b \rangle + (r_i)_j. \tag{1}$$

Also, by the definition of the bracket, we have

$$a_j \geq \langle r_i \mid a \rangle + (r_i)_j. \tag{2}$$

Since all terms in (1) and (2) (with the possible exception of a_j which may be ∞) are known to be finite, we may subtract (1) from (2) to obtain

$$a_j - b_j \geq \langle r_i \mid a \rangle + (r_i)_j - (r_i)_j - \langle r_i \mid b \rangle = \langle r_i \mid a \rangle - \langle r_a \mid b \rangle = x_i - y_i.$$

But now by Proposition 1.2 we have

$$\langle a \mid b \rangle = -\bigoplus_{k=1}^{n} \{a_k(-b_k)\} \leq -(a_j - b_j) \leq -(x_i - y_i) = \langle x \mid y \rangle.$$

It remains to show that $\langle x \mid y \rangle \leq \langle a \mid b \rangle$ under the assumption that a and b lie in the convex hull of the vectors $r_1, \ldots, r_k$. Under this assumption,

Proposition 1.5 ensures that

$$a = \bigoplus_{i=1}^{k} \langle r_i \mid a \rangle r_i \text{ and } b = \bigoplus_{i=1}^{k} \langle r_i \mid b \rangle r_i.$$

Suppose $\lambda \in \overline{\mathbb{T}}$ is such that $\lambda x \leq y$. Then by definition $\lambda x_i \leq y_i$ for all i, that is, $\lambda \langle r_i \mid a \rangle \leq \langle r_i \mid b \rangle$ for all i. Using the compatibility of the order with $\otimes$ it follows that $\lambda \langle r_i \mid a \rangle r_i \leq \langle r_i \mid b \rangle r_i$ for all i and hence using the compatibility of the order with $\oplus$ and distributivity of $\otimes$ over $\oplus$ that

$$\lambda a = \lambda \bigoplus_{i=1}^{k} \langle r_i \mid a \rangle r_i = \bigoplus_{i=1}^{k} \lambda \langle r_i \mid a \rangle r_i \leq \bigoplus_{i=1}^{k} \langle r_i \mid b \rangle r_i = b.$$

We have shown that $\{\lambda \mid \lambda x \leq y\} \subseteq \{\lambda \mid \lambda a \leq b\}$ and so

$$\langle x \mid y \rangle = \max\{\lambda \mid \lambda x \leq y\} \leq \max\{\lambda \mid \lambda a \leq b\} = \langle a \mid b \rangle.$$

□

Lemma 2.2 is the key ingredient for our new formulation of tropical matrix duality:

Let $X \subseteq \overline{\mathbb{T}}^n$ and $Y \subseteq \overline{\mathbb{T}}^m$ be convex sets. We say that a function $\theta : X \to Y$ is an *anti-morphism* if

- for all $x, y \in X$, we have $\langle x \mid y \rangle = \langle \theta(y) \mid \theta(x) \rangle$; and
- for all $x \in X$ and $\lambda \in \mathbb{FT}^n$ we have $\theta(\lambda x) = (-\lambda)\theta(x)$.

Notice that an anti-morphism is required to preserve scaling only by *finite* scalars and that, by Proposition 1.4, an anti-morphism must be order-reversing. A bijective anti-morphism is called an *anti-isomorphism*; an anti-isomorphism is in particular an antitone lattice morphism. Notice that the inverse of an anti-isomorphism is necessarily an anti-isomorphism. Two sets are termed *anti-isomorphic* if there is an anti-isomorphism between them.

Lemma 2.3. *Let $S = \overline{\mathbb{T}}$ or $S = \mathbb{FT}$, let $X \subseteq S^i$, $Y \subseteq S^j$ and $Z \subseteq S^k$ be convex sets, and suppose $\theta_1 : X \to Y$ and $\theta_2 : Y \to Z$ are anti-isomorphisms between convex sets. Then the composition $\theta_2 \circ \theta_1$ is a linear isomorphism of semimodules.*

Proof. Since θ_1 and θ_2 are antitone lattice isomorphisms, their composition is certainly a lattice isomorphism. The fact that $\theta_2 \circ \theta_1$ respects addition now follows from the fact that addition can be defined in terms of the lattice order in X and Z. Indeed, since X is convex, for any elements $x, y \in X$, $x \oplus y$ is the least upper bound of x of y in the lattice order on X. Since $\theta_2 \circ \theta_1$ is a lattice isomorphism it follows that $\theta_2(\theta_1(x \oplus y))$ is the least upper bound of $\theta_2(\theta_1(x))$ and $\theta_2(\theta_1(y))$ in the lattice order on Z, which since Z is convex is exactly $\theta_2(\theta_1(x)) \oplus \theta_2(\theta_1(y))$.

Also, for any finite λ we have

$$\theta_2(\theta_1(\lambda x)) = \theta_2((-\lambda)\theta_1(x)) = \lambda\theta_2(\theta_1(x))$$

so the composition $\theta_2 \circ \theta_1$ preserves scaling by finite scalars. In the case $S = \mathbb{FT}$, this completes the proof.

In the case $S = \overline{\mathbb{T}}$, we must show also that $\theta_2 \circ \theta_1$ preserves scaling by $-\infty$ and ∞. Let z_i and z_k denote the zero vectors in $\overline{\mathbb{T}}^i$ and $\overline{\mathbb{T}}^k$ respectively. Since X, Y and Z are convex, each contains the zero vector of the appropriate dimension. Indeed, each must have the zero vector as its bottom element, which since $\theta_2 \circ \theta_1$ is a lattice isomorphism means that $\theta_2(\theta_1(z_i)) = z_k$. Thus we have

$$\theta_2(\theta_1((-\infty)x)) = \theta_2(\theta_1(z_i)) = z_k = (-\infty)\theta_2(\theta_1(x))).$$

It remains only to show that $\theta_2 \circ \theta_1$ preserves scaling by ∞. Notice that for any vector $x \in \overline{\mathbb{T}}^n$ we have

$$\infty x = \sup\{\lambda x \mid \lambda \in \mathbb{FT}\}.$$

The supremum here is by definition taken in $\overline{\mathbb{T}}^n$, but if x lies in some convex set $S \subseteq \overline{\mathbb{T}}^n$ then ∞x and λx for all $\lambda \in \mathbb{FT}$ also lies in S, and it follows that we may take the supremum in S. Now since $\theta_2 \circ \theta_1$ is a lattice isomorphism of convex sets, it preserves suprema within X. Since it also preserves scaling by finite scalars, we have:

$$\begin{aligned}
\theta_2(\theta_1(\infty x)) &= \theta_2(\theta_1(\sup\{\lambda x \mid \lambda \in \mathbb{FT}\})) \\
&= \sup\{\theta_2(\theta_1(\lambda x)) \mid \lambda \in \mathbb{FT}\} \\
&= \sup\{\lambda \theta_2(\theta_1(x)) \mid \lambda \in \mathbb{FT}\} \\
&= \infty \theta_2(\theta_1(x)).
\end{aligned}$$

□

Theorem 2.4 (Algebraic Duality Theorem)**.** *Let A be a matrix over $\overline{\mathbb{T}}$. Then the map $\theta_A : R(A) \to C(A)$ is an anti-isomorphism between $R(A)$ and $C(A)$. If all entries of A are finite then θ_A restricts to an anti-isomorphism between $R_{\mathbb{FT}}(A)$ and $C_{\mathbb{FT}}(A)$.*

Proof. By Proposition 2.1, θ_A is a bijection from $R(A)$ to $C(A)$. Suppose $a, b \in R(A)$, and let $A_1, \dots, A_p$ be the rows of A. Then by definition, a and b lie in the convex hull of the A_i's. Using the definition of θ_A, Lemma 2.2 says exactly that $\langle a \mid b\rangle = \langle -\theta_A(a) \mid -\theta_A(b)\rangle$. But by Proposition 1.3 we have $\langle -\theta_A(a) \mid -\theta_A(b)\rangle = \langle \theta_A(b) \mid \theta_A(a)\rangle$.

Next, suppose the columns of A are $B_1, \dots, B_q$ and let $x \in R(A)$ and $\lambda \in \mathbb{FT}$. Then $-\lambda \in \mathbb{FT}$ and we have

$$\theta_A(\lambda x) = \bigoplus_{i=1}^{q}(-(\lambda x)_i)B_i = \bigoplus_{i=1}^{q}(-\lambda)(-x_i)B_i = (-\lambda)\bigoplus_{i=1}^{q}(-x_i)B_i = (-\lambda)\theta_A(x).$$

□

We shall see later (Theorem 4.4 below) that the existence of an anti-isomorphism between finitely generated convex sets X and Y is a sufficient, as well as a necessary, condition for X and Y to be the row space and column space of a matrix over $\mathbb{FT}$ or $\overline{\mathbb{T}}$.

As a corollary of Theorem 2.4, we obtain a special case of the theorem of Cohen, Gaubert and Quadrat [10].

Theorem 2.5 (Lattice Duality Theorem [10])**.** *Let A be a matrix over $\overline{\mathbb{T}}$. Then the duality maps θ_A and θ'_A are mutually inverse antitone lattice isomorphisms between $R(A)$ and $C(A)$. If A has all entries finite then θ_A and θ'_A restrict to mutually inverse antitone lattice isomorphisms between $R_{\mathbb{FT}}(A)$ and $C_{\mathbb{FT}}(A)$.*

Proof. By Proposition 2.1, θ_A and θ'_A are mutually inverse bijections between $R(A)$ and $C(A)$, and restrict to mutually inverse bijections between $R_{\mathbb{FT}}(A)$ and $C_{\mathbb{FT}}(A)$ where appropriate. Hence, it will suffice to show that θ_A is order-reversing. For any $x, y \in R(A)$, by Theorem 2.4 we have $\langle x \mid y \rangle = \langle \theta_A(y) \mid \theta_A(x) \rangle$. Thus, using Proposition 1.4 twice, $x \leq y$ if and only if $\langle x \mid y \rangle = \langle \theta_A(y) \mid \theta_A(x) \rangle \geq 0$, which holds exactly if $\theta_A(y) \leq \theta_A(x)$. □

Another immediate consequence of Theorem 2.4, which does not seem to have been previously noted, is that the duality map induces an isometry (with respect to the Hilbert metric) between the projective row space and projective column space of a tropical matrix.

Theorem 2.6 (Metric Duality Theorem)**.** *Let A be a tropical matrix. Then the duality maps θ_A and θ'_A induce mutually inverse isometries (with respect to the Hilbert projective metric) between the projectivisations of $R(A)$ and $C(A)$. If the entries of A are all finite then their restrictions induce mutually inverse isometries (with respect to the Hilbert projective metric) between the projectivisations of $R_{\mathbb{FT}}(A)$ and $C_{\mathbb{FT}}(A)$.*

Proof. By Proposition 2.1, θ_A and θ'_A are mutually inverse bijections between $R(A)$ and $C(A)$. By Theorem 2.4 they map finite scalings to finite scalings, and hence induce well-defined maps on the respective projective spaces. Also by Theorem 2.4, they reverse the bracket operation, and hence by the observations at the end of Section 1, they preserves the distance function. □

3. Green's Relations

In this section we briefly recall the definitions of Green's relations; for a more detailed introduction we refer the reader to one of the introductory texts on semigroup theoery, such as [8]. We then prove some foundational results about Green's relations in tropical matrix semigroups.

Let S be any semigroup. We denote that S^1 the monoid obtained by adjoining an extra identity element 1 to S. We define a pre-order (a reflexive, transitive binary relation) $\leq_{\mathscr{R}}$ on S by $a \leq_{\mathscr{R}} b$ if there exists $c \in S^1$ such that $bc = a$, that is, if a lies in the principle right ideal bS^1 generated by b, or equivalently, if $aS^1 \subseteq bS^1$. We define an equivalence relation $\mathscr{R}$ on S by $a \mathscr{R} b$ if $a \leq b$ and $b \leq a$, that is, if $aS^1 = bS^1$. The pre-order $\leq_{\mathscr{R}}$ thus induces a partial order on the set of equivalence classes of $\mathscr{R}$.

Dually, we define $a \leq_{\mathscr{L}} b$ if $S^1 a \subseteq S^1 b$, and $a \mathscr{L} b$ if $S^1 a = S^1 b$. Similarly, we let $a \leq_{\mathscr{J}} b$ if $S^1 a S^1 \subseteq S^1 b S^1$ and $a \mathscr{J} b$ if $S^1 a S^1 = S^1 b S^1$. We let $\mathscr{H}$ be the intersection of $\mathscr{L}$ and $\mathscr{R}$ (so $a \mathscr{H} b$ if $a \mathscr{L} b$ and $a \mathscr{R} b$) and $\mathscr{D}$ be the smallest equivalence relation containing both $\mathscr{L}$ and $\mathscr{R}$. It is well known and easy to show (see for example [8]) that $a \mathscr{D} b$ if and only if there exists $c \in S$ with $a \mathscr{L} c$ and $c \mathscr{R} b$ (or dually, if and only if there exists $d \in S$ with $a \mathscr{R} d$ and $d \mathscr{L} b$).

We begin with an elementary description of the relations $\mathscr{R}$ and $\mathscr{L}$ for full matrix semigroups over our tropical semirings, and indeed over a wider class of semirings. We say that a commutative semiring S has *local zeros* if for every finite set $X \subseteq S$ there exists an element $z \in S$ such that $z + x = x$ for all $x \in X$. The semirings $\mathbb{FT}$, $\mathbb{T}^{n\times n}$ and $\overline{\mathbb{T}}$ all have local zeros, since given any finite set X if elements it suffices to choose z to be any element smaller than those in X. (In the latter two cases one may of course always choose ∞). The following is a generalisation of a result proved for $\mathbb{T}^{n\times n}$ in [23].

Proposition 3.1. *Let A and B be elements of a full matrix semigroup over a commutative semiring with multiplicative identity and local zeros. Then*

(i) *$A \leq_R B$ if and only if $C(A) \subseteq C(B)$;*
(ii) *$A \mathscr{R} B$ if and only if $C(A) = C(B)$;*
(iii) *$A \leq_L B$ if and only if $R(A) \subseteq R(B)$;*
(iv) *$A \mathscr{L} B$ if and only if $R(A) = R(B)$;*
(v) *$A \mathscr{H} B$ if and only if $C(A) = C(B)$ and $R(A) = R(B)$.*

Proof. It clearly suffices to show (i). Indeed, (iii) is dual to (i), (ii) and (iv) follow from (i) and (ii) respectively, and (v) follows from (ii) and (iv).

Suppose, then, that $A \leq_{\mathscr{R}} B$, then by definition there is a matrix $X \in S^{n\times n}$ such that $BX = A$. Now, since the columns of BX are contained in $C(B)$ it follows that $C(BX) = C(A) \subseteq C(B)$.

Conversely, suppose $C(A) \subseteq C(B)$. Since the semiring has a multiplicative identity, every column of A is a linear combination of a subset of the columns of A, and hence lies in $C(A)$ and also in $C(B)$. Thus, every column of A can be written as a linear combination of some subset of the columns of B. But since the semiring has local zeros, this means it can be written as a linear combination of all of the columns of B, which means exactly that there exists $X \in S^{n\times n}$ such that $A = BX$, and so $A \leq_{\mathscr{R}} B$. $\square$

We next consider the relationship between Green's relations in the respective full matrix semigroups over the three semirings $\mathbb{FT}$, $\mathbb{T}$ and $\overline{\mathbb{T}}$. Notice that if A is a matrix over one semiring which is contained in another, then the row and column space of A depend upon the semiring over which it is considered. Hence, it is not immediate from Proposition 3.1 that, for example, two matrices in $\mathbb{FT}^{n\times n}$ which are $\mathscr{R}$-related in $\overline{\mathbb{T}}^{n\times n}$ must also $\mathscr{R}$-related in $\mathbb{FT}^{n\times n}$. However, it transpires that this is nevertheless the case.

Proposition 3.2. *Let A and B be matrices in $\mathbb{FT}^{n\times n}$. Then $A \leq_{\mathscr{R}} B$ in $\mathbb{FT}^{n\times n}$ if and only if $A \leq_{\mathscr{R}} B$ in $\mathbb{T}^{n\times n}$.*

Proof. Suppose $A \leq_{\mathscr{R}} B$ in $\mathbb{T}^{n\times n}$. Then there is a matrix $P \in \mathbb{T}^{n\times n}$ such that $A = BP$. Since B and P have finitely many entries, we may choose some finite $\delta \in \mathbb{FT}$ smaller than $b + p - b'$ for every pair of entries b, b' of B and every finite entry p of P. Let P' be obtained from P by replacing every $-\infty$ entry with δ.

Now let $1 \leq i, j \leq n$. Then

$$(BP')_{ij} = \bigoplus_{k=1}^{n} B_{ik} P'_{kj} \tag{3}$$

while

$$A_{ij} = (BP)_{ij} = \bigoplus_{k=1}^{n} B_{ik}P_{kj}. \tag{4}$$

Choose k such that $B_{ik}P_{kj}$ is maximum, that is, such that $B_{ik}P_{kj} = A_{ij}$. Since A_{ij} is finite, we must have P_{kj} finite, so $P'_{kj} = P_{kj}$ and $B_{ik}P'_{kj} = B_{ik}P_{kj} = A_{ij}$. Moreover, all the other entries in the maximum in (3) are either of the form $B_{ih}P_{hj}$ (which cannot exceed $B_{ik}P_{kj}$ by the assumption on k) or of the form $B_{ih}\delta$ (which cannot exceed $B_{ik}P_{kj}$ by the definition of δ). Thus, the maximum in (3) is A_{ij} and we have $A = BP'$.

The converse is immediate. □

Proposition 3.3. *Let A and B be matrices in $\mathbb{T}^{n\times n}$. Then $A \leq_{\mathscr{R}} B$ in $\mathbb{T}^{n\times n}$ if and only if $A \leq_{\mathscr{R}} B$ in $\overline{\mathbb{T}}^{n\times n}$.*

Proof. Suppose $A \leq_{\mathscr{R}} B$ in $\overline{\mathbb{T}}^{n\times n}$. Then there is a matrix $P \in \overline{\mathbb{T}}^{n\times n}$ such that $A = BP$. Let $P' \in \mathbb{T}^{n\times n}$ be obtained from P by replacing every ∞ entry with 0 (or indeed any other element of $\mathbb{T}$). Now for $1 \leq i, j \leq n$ we have

$$A_{ij} = (BP)_{ij} = \bigoplus_{k=1}^{n} B_{ik}P_{kj}. \tag{5}$$

Since no A_{ij} is ∞, if any k and j are such that $P_{kj} = \infty$ then we must have $B_{ik} = -\infty$ for all i. It follows that $B_{ik}P_{kj} = -\infty = B_{ik}P'_{kj}$ for all k and j, whence $A = BP'$ and $A \leq_{\mathscr{R}} B$ in $\mathbb{T}^{n\times n}$.

Again, the converse is immediate. □

Lemma 3.4. *Suppose $X \in \mathbb{T}^{n\times n}$ is $\mathscr{R}$-related to a matrix in $\mathbb{FT}^{n\times n}$, and $\mathscr{L}$-related to a matrix in $\mathbb{FT}^{n\times n}$. Then $X \in \mathbb{FT}^{n\times n}$.*

Proof. Suppose $X\mathscr{R}Y \in \mathbb{FT}^{n\times n}$. Then any column of X containing $-\infty$ lies in $C(X)$, which by Proposition 3.1 is $C(Y)$. But it is easily seen that the only column vector in $C(Y)$ containing $-\infty$ is the zero vector, so every column of X containing $-\infty$ is a column of $-\infty$s. A dual argument, using the fact that X is $\mathscr{L}$-related to a matrix in $\mathbb{FT}^{n\times n}$, shows that every row of X containing $-\infty$ is a row of $-\infty$s. Now if $X \notin \mathbb{FT}^{n\times n}$ then X contains some entry equal to $-\infty$, from which we may deduce that every entry of X is $-\infty$, that is, that X is the zero matrix. But the zero matrix forms an ideal, and so must lie in a $\mathscr{R}$-class by itself, contradicting the fact that $X\mathscr{R}Y$. □

Lemma 3.5. *Suppose $X \in \overline{\mathbb{T}}^{n\times n}$ is $\mathscr{R}$-related to a matrix in $\mathbb{T}^{n\times n}$, and $\mathscr{L}$-related to a matrix in $\mathbb{T}^{n\times n}$. Then $X \in \mathbb{T}^{n\times n}$.*

Proof. We claim first that the column space $C(X)$ of X is generated by those columns which do not contain ∞. Indeed, suppose $X\mathscr{R}Y \in \mathbb{T}^{n\times n}$. Then by Proposition 3.1, $C(X) = C(Y)$. In particular, each column of Y is a linear combination of columns of X. Clearly this combination cannot a column of X with an ∞ entry with a coefficient other than $-\infty$, or else the column of Y with contain ∞. Thus, each column of Y is a linear combination of those columns of X which do not contain ∞. But every vector in $C(X) = C(Y)$ is a linear

combination of the columns of Y, and hence of the columns of X which do not contain ∞, as required.

By a dual argument, the row space of X is generated by those rows which do not contain ∞.

Now suppose for a contradiction that $X \notin \mathbb{T}^{n\times n}$, and choose some row i and column j with $X_{ij} = \infty$. By the above, the jth column (call it X_j) can be written as a linear combination of those columns not containing ∞. Clearly, one of the columns in this combination (say column X_k) must have coefficient ∞ and a finite entry in position i. Now $\infty X_k \leq X_j$, so for any p such that $X_{pj} \neq \infty$ we must have $X_{pk} = -\infty$. In particular, in any row of X not containing ∞, column k will contain $-\infty$. Since the rows not containing ∞ span the row space $R(X)$, it follows that every row vector in $R(X)$ contains $-\infty$ in column k. But since the rows of X lie in $R(X)$, this contradicts the fact that row i of X contains a finite entry in column k. □

The preceding propositions and lemmas combine to show that many of Green's relations in $\mathbb{FT}^{n\times n}$ and $\mathbb{T}^{n\times n}$ are inherited from the containing semigroup $\overline{\mathbb{T}}^{n\times n}$.

Theorem 3.6 (Inheritance of Green's Relations)**.** *Each of Green's pre-orders $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$ and equivalence relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ in $\mathbb{FT}^{n\times n}$ or $\mathbb{T}^{n\times n}$ is the restriction of the corresponding relation in $\overline{\mathbb{T}}^{n\times n}$.*

Proof. The results for $\mathscr{R}$ and $\mathscr{L}$ follow immediately from Propositions 3.2 and 3.3 and their duals. The claim for $\mathscr{H}$ is immediate from the claims for $\mathscr{R}$ and $\mathscr{L}$. The claim for $\mathscr{D}$ is a consequence of the claims for $\mathscr{R}$ and $\mathscr{L}$ together with Lemma 3.4 and Lemma 3.5. □

We also have the following immedate corollary of Lemmas 3.4 and 3.5.

Corollary 3.7. *$\mathbb{FT}^{n\times n}$ is a union of $\mathscr{H}$-classes in $\mathbb{T}^{n\times n}$ and in $\overline{\mathbb{T}}^{n\times n}$, while $\mathbb{T}^{n\times n}$ is a union of $\mathscr{H}$-classes in $\overline{\mathbb{T}}^{n\times n}$.*

4. Converse Duality

In this section we shall establish a converse to Theorem 2.4 in the finitary case, showing that an anti-isomorphism between two convex sets X and Y in $\mathbb{FT}^n$ is a sufficient, as well as a necessary, condition for the existence of a matrix with row space X and column space Y. As well as being of interest in its own right, this together with Theorem 3.6 will allow us to completely describe Green's $\mathscr{D}$ relation in finite dimensional full matrix semigroups over the tropical semirings $\mathbb{FT}$, $\mathbb{T}$ and $\overline{\mathbb{T}}$. We begin with some lemmas.

Lemma 4.1. *Let $S = \mathbb{FT}$ or $S = \overline{\mathbb{T}}$. Suppose $B \in S^{m\times n}$ is a tropical matrix and $z \in S^{1\times n}$ is a row vector not in $R_S(B)$. Then there exist column vectors $x, y \in S^{n\times 1}$ such that $Bx = By$ but $zx \neq zy$.*

Proof. Set $x = (-z)^T$, and consider the vector Bx, which clearly lies in the column space of B. By Proposition 2.1 the map θ_B is a bijection from the $R_S(B)$ to $C_S(B)$, so there is a $v \in R_S(B)$ such that $\theta_B(v) = Bx$. Note that $v \neq z$ since z does not lie in $R_S(B)$. If we set $y = (-v)^T$ then by the definition

of θ_B we have $By = B(-v)^T = \theta_B(v) = Bx$, so it will suffice to show that $zx \neq zy$.

To this end, consider the matrix

$$C = \begin{pmatrix} z \\ B \end{pmatrix} \in \overline{\mathbb{T}}^{(m+1)\times n}.$$

Then z (which is a row of C) and v (which was chosen to lie in $R_S(B)$) both lie in $R_S(C)$. Consider now the duality map θ_C. By Proposition 2.1 again, θ_C is injective on $R_S(C)$, so we have

$$Cx = C(-z)^T = \theta_C(z) \neq \theta_C(v) = C(-v)^T = Cy.$$

But

$$Cx = \begin{pmatrix} z \\ B \end{pmatrix} x = \begin{pmatrix} zx \\ Bx \end{pmatrix} \text{ and } Cy = \begin{pmatrix} z \\ B \end{pmatrix} y = \begin{pmatrix} zy \\ By \end{pmatrix}$$

and we know that $Bx = By$, so for $Cx \neq Cy$ we must have $zx \neq zy$. □

Theorem 4.2. *Let $S = \mathbb{FT}$ or $S = \overline{\mathbb{T}}$, and let $A, B \in S^{m\times n}$. Then the following are equivalent:*

(i) $R_S(A) \subseteq R_S(B)$.
(ii) *there is a linear morphism from $C_S(B)$ to $C_S(A)$ taking the ith column of B to the ith column of A for all i.*
(iii) *there is a surjective linear morphism from $C_S(B)$ to $C_S(A)$ taking the ith column of B to the ith column of A for all i.*

Proof. First note that (iii) implies (ii) trivially, while if (ii) holds then the given morphism has image including the columns of A, and hence contains $C_S(A)$, and thus is surjective, so (iii) holds.

Now let $c_1, \dots, c_n$ denote the columns of A and $d_1, \dots, d_n$ denote the columns of B.

Suppose for a contradiction that (ii) holds and (i) does not. Then we may choose $z \in R_S(A)$ (say $z = z'A$) such that $z \notin R_S(B)$. Now by Lemma 4.1, there are vectors x and y such that $Bx = By$ but $zx \neq zy$. It follows from the latter that $Ax \neq Ay$, since otherwise we would have $zx = z'Ax = z'Ay = zy$. Now by the definition of matrix multiplication we have

$$\bigoplus_{i=1}^{n} x_i c_i = Ax \neq Ay = \bigoplus_{i=1}^{n} y_i c_i$$

while

$$\bigoplus_{i=1}^{n} x_i d_i = Bx = By = \bigoplus_{i=1}^{n} y_i d_i,$$

which clearly contradicts the assumption that the map taking d_i to c_i is a morphism of semimodules.

Conversely, suppose (i) holds. To show that (ii) holds it clearly suffices to show that every linear relation between the columns of B also holds between the columns of A. Indeed, suppose

$$\bigoplus_{i=1}^{n} x_i c_i = \bigoplus_{i=1}^{n} y_i c_i$$

is a relation which holds between the columns c_i of A. Then letting x and y be the column vectors formed from the x_is, by the definition of matrix multiplication we have $Bx = By$. It follows that $bx = by$ for every row b of B, and hence by distributivity for every vector in $R_S(B)$. In particular, $bx = by$ for every vector in $R_S(A) \subseteq R_S(B)$, so that $Ax = Ay$ and

$$\bigoplus x_i d_i = \bigoplus y_i d_i$$

as required. □

Corollary 4.3. *Let $S = \overline{\mathbb{T}}$ or $S = \mathbb{FT}$, and let $A, B \in S^{m \times n}$. Then $R_S(A) = R_S(B)$ if and only if there is a linear isomorphism from $C_S(A)$ to $C_S(B)$ taking the ith column of B to the ith column of A for all i.*

Proof. If $R_S(A) = R_S(B)$ then $R_S(A) \subseteq R_S(B)$ and $R_S(B) \subseteq R_S(A)$, so by applying Theorem 4.2 twice there is a surjective morphism from $C_S(B)$ to $C_S(A)$ taking the columns of B to the respective columsn of A, and a surjective morphism from $C_S(A)$ to $C_S(B)$ taking the columns of A to the respective columns of $C_S(B)$. Since these maps are mutually inverse on the columns, which are generating sets for the respective matrices, it is immediate that they are mutually inverse maps from $C_S(A)$ to $C_S(B)$, and hence must be isomorphisms.

Conversely, if $f : C_S(A) \to C_S(B)$ is an isomorphism taking the columns of A to the respective columns of B, then its inverse is a morphism taking the columns of B to the respective columns of A. Applying Theorem 4.2 to each of these functions we obtain $R_S(A) \subseteq R_S(B)$ and $R_S(B) \subseteq R_S(A)$. □

The above results allow us to establish our promised converse to the duality theorem (Theorem 2.4 above).

Theorem 4.4 (Exact Duality Theorem)**.** *Let $S = \mathbb{FT}$ or $S = \overline{\mathbb{T}}$. Suppose X be an m-generated convex subset of S^n and Y is an n-generated convex subset of S^m. Then X and Y are anti-isomorphic if and only if there is a matrix $M \in S^{m \times n}$ with $R_S(M) = X$ and $C_S(M) = Y$.*

Proof. Suppose X and Y are anti-isomorphic. Choose two $m \times n$ matrices A and B such that A has row space X and B has column space Y. Then by the dual to Theorem 2.4, there is an anti-isomorphism from $C_S(A)$ to $R_S(A) = X$. By Lemma 2.3, composing with the anti-isomorphism from X and $Y = C_S(B)$ we may thus obtain an isomorphism $f : C_S(A) \to C_S(B)$. Let D be the $m \times n$ matrix whose ith column is the image under f of the ith column of A. Then by Corollary 4.3 we have $R_S(A) = R_S(D)$. Also, since f is an isomorphism, the image under f of a generating set for $C_S(A)$ must be a generating set for $C_S(B)$. In particular, the columns of D are a generating set for $C_S(B)$, that is, $C(D) = C(B) = Y$. Now by Corollary 4.2 to f and its inverse, we have $R_S(D) \subseteq R_S(A) = X$ and $X = R_S(A) \subseteq R_S(D)$. Thus, the matrix D has the required properties.

The converse is Theorem 2.4. □

5. The $\mathscr{D}$ Relation

From Theorem 4.4 and Lemma 2.3 we obtain a number of equivalence geometric characterisations of Green's $\mathscr{D}$ relation in $\mathbb{FT}$ and $\overline{\mathbb{T}}$.

Theorem 5.1 (Green's $\mathscr{D}$ Relation for $\mathbb{FT}^{n\times n}$ and $\overline{\mathbb{T}}^{n\times n}$)**.** *Let $S = \mathbb{FT}$ or $S = \overline{\mathbb{T}}$, and let A and B be matrices in $S^{n\times n}$. Then the following are equivalent:*

(i) *$A\mathscr{D}B$ in $S^{n\times n}$;*
(ii) *$C_S(A)$ and $C_S(B)$ are isomorphic as semimodules;*
(iii) *$R_S(A)$ and $R_S(B)$ are isomorphic as semimodules;*
(iv) *$C_S(A)$ and $R_S(B)$ are anti-isomorphic as semimodules;*
(v) *$R_S(A)$ and $C_S(B)$ are anti-isomorphic as semimodules.*

Proof. First suppose (i) holds. Then by definition there exists a matrix D such that $A\mathscr{L}D\mathscr{R}B$. By Proposition 3.1, we have $R_S(A) = R_S(D)$ and $C_S(D) = C_S(B)$. But by Theorem 4.4, $R_S(D)$ and $C_S(D)$ are anti-isomorphic, so $R_S(A)$ and $C_S(B)$ are anti-isomorphic, and so (v) holds. A dual argument shows that (i) implies (iv).

Next suppose (v) holds. By Theorem 4.4, there is an anti-isomorphism from $C_S(A)$ to $R_S(A)$. By Lemma 2.3 this composes with the anti-isomorphism from $R_S(A)$ to $C_S(B)$ to produce an isomorphism between $C_S(A)$ and $C_S(B)$, so that (ii) holds. Similar arguments establish that (v) implies (iii), (iv) implies (iii) and (iv) implies (ii).

Finally, suppose (ii) holds, and let $f : C_S(A) \to C_S(B)$ be an isomorphism. Let D be the matrix obtained from A by applying f to each column. Then by Corollary 4.3, we have $R_S(A) = R_S(D)$ so that $A\mathscr{L}D$. Moreover, since the isomorphism f must map a generating set for $C_S(A)$ to a generating set for $C_S(B)$, the columns of D form a generating set for $C_S(D)$, that is, $C_S(D) = C_S(B)$, so $D\mathscr{R}B$. Thus, $A\mathscr{D}B$ and (i) holds. □

Theorem 5.1 and Theorem 3.6 together yield a description of $\mathscr{D}$ for matrices over $\mathbb{T}$ in terms of their $\overline{\mathbb{T}}$-linear column or row spaces. It is natural to ask also whether $\mathscr{D}$ can be characterised in terms of $\mathbb{T}$-linear column and row spaces.

Lemma 5.2. *Let X be a convex subset of $\mathbb{T}^n$ and X' be the convex subset of $\overline{\mathbb{T}}^n$ which it generates. Then for any $x \in X'$ the following are equivalent.*

(i) *$x \notin X$;*
(ii) *x contains ∞ in some component;*
(iii) *$x = \infty a \oplus b$ for some $a, b \in X'$ with a not the zero vector;*
(iv) *$x = \infty a \oplus b$ for some $a, b \in X$ with a not the zero vector;*

Proof. Suppose (i) holds, that is, that $x \notin X$. Since $x \in X'$, it may be written as a $\overline{\mathbb{T}}$-linear combination of finitely many vectors in X. Using distributivity and commutativity to collect together the terms with coefficient ∞ and the terms with other coefficients, we may thus write $x = \infty a \oplus b$ where a is a sum of vectors in X (and hence lies in X), and b is a $\mathbb{T}$-linear combination of vectors in X (and hence lies in X). Finally, if a were the zero vector then we would have $x = b \in X$ giving a contradiction. Thus, (iv) holds.

That (iv) implies (iii) is immediate. If (iii) holds then since a is not the zero vector, ∞a contains ∞ in some component, so $x = \infty a \oplus b$ contains ∞ in some component, and (ii) holds. Finally, that (ii) implies (i) is obvious. □

Lemma 5.3. *Let X be a convex subset of $\mathbb{T}^n$ and X' be the convex subset of $\overline{\mathbb{T}}^n$ which it generates. Then for any $a, b, a', b' \in X$ we have that $\infty a \oplus b = \infty a' \oplus b'$ if and only if $d_H(a, a') \neq \infty$ and $b \oplus \lambda a = b' \oplus \lambda a$ for all sufficiently large λ.*

Proof. Suppose $\infty a \oplus b = \infty a' \oplus b'$. Since $a, b \in \mathbb{T}^n$, they do not contain any ∞ positions. It follows that $\infty a \oplus b$ contains an ∞ in position i exactly if a does **not** contain $-\infty$ in this position. By symmetry of assumption this is true exactly if a' does **not** contain $-\infty$ in this position. Thus, a and a' contain $-\infty$ in exactly the same positions. Since neither contains ∞, this means that $d_H(a, a') \neq \infty$. Notice also that in any position where a contains $-\infty$, the expression $\infty a \oplus b$ takes the value of b and hence by symmetry also of b'. Thus, b and b' agree in such positions. Hence, if we choose λ large enough that λa exceeds b and b' in all positions where a is not $-\infty$, then we obtain $b \oplus \lambda a = b' \oplus \lambda a$.

Conversely, suppose $d_H(a, a') \neq \infty$ and $b \oplus \lambda a = b' \oplus \lambda a$ for all sufficiently large λ. Then a and a' have $-\infty$ in the same positions, from which it follows that $\infty a = \infty a'$. Moreover, since $b \oplus \lambda a = b' \oplus \lambda a$ it is easy to see that b and b' agree in every position where a takes the value $-\infty$, from which it follows that $\infty a \oplus b = \infty a \oplus b' = \infty a' \oplus b'$. □

Theorem 5.4 (Inheritance and Extension of Isomorphisms)**.** *Let X and Y be convex subsets of $\mathbb{T}^i$ and $\mathbb{T}^j$ respectively, and let X' and Y' be the convex subsets of $\overline{\mathbb{T}}^i$ and $\overline{\mathbb{T}}^j$ which they generate. Then X and Y are isomorphic (as semimodules over $\mathbb{T}$) if and only if X' and Y' are isomorphic (as semimodules over $\overline{\mathbb{T}}$).*

Proof. Suppose first that $f : X' \to Y'$ is an isomorphism. We claim that f sends elements of X to elements of Y. Indeed, suppose $x \in X$. Then by Lemma 5.2, x cannot be written in the form $a\infty \oplus b$ for any $a, b \in X'$ with a not the zero vector. Since f is an isomorphism (and in particular preserves the zero vector) it follows that $f(x)$ cannot be written as $\infty c \oplus d$ for any $c, d \in Y'$ with c not the zero vector. Thus, by Lemma 5.2 again, $f(x)$ lies in Y. A similar argument shows that the inverse of f maps Y into X, and it follows that f restricts to an isomorphism of X to Y.

Conversely, suppose that $g : X \to Y$ is an isomorphism. We claim that g admits an extension to X' well defined by:

$$\hat{g}(\infty a \oplus b) = \infty g(a) \oplus g(b).$$

To show that this is well defined, suppose $\infty a \oplus b = \infty a' \oplus b'$. Then by Lemma 5.3 we have $d_H(a, a') \neq \infty$ and $b \oplus \lambda a = b' \oplus \lambda a$ for all sufficiently large λ. Using the fact that g is an isomorphism (and in particular preserves the Hilbert metric) we have $d_H(g(a), g(a')) \neq \infty$ and $g(b) \oplus \lambda g(a) = g(b') \oplus \lambda a$ for all sufficiently large λ. Now by Lemma 5.3 again, $\infty g(a) \oplus g(b) = \infty g(a') \oplus g(b')$, as required to show that $\hat{g}$ is well-defined.

Next we claim that $\hat{g}$ is linear. The fact that $\hat{g}$ respects addition and scaling by elements of $\mathbb{T}$ follows immediately from the definition and the elementary properties of the semiring $\overline{\mathbb{T}}$. It remains to show that $\hat{g}$ respects scaling by ∞.

Let $x \in X'$. Then x can be written as $\infty a \oplus b$ for some $a, b \in X$, and we have

$$\begin{aligned}\hat{g}(\infty x) &= \hat{g}(\infty(\infty a \oplus b)) = \hat{g}(\infty a \oplus \infty b) = \infty g(a \oplus b) = \infty g(a) \oplus \infty g(b) \\ &= \infty\infty g(a) \oplus \infty g(b) = \infty(\infty g(a) \oplus g(b)) = \infty \hat{g}(\infty a \oplus b) = \infty \hat{g}(x).\end{aligned}$$

Now if $h : Y \to X$ is the inverse of g then the same argument shows that h extends to a linear map $\hat{h} : Y' \to X'$ satisfying $\hat{h}(\infty c \oplus d) = \infty h(c) \oplus h(d)$ for all $c, d \in Y$. Thus, for any $x \in X'$ we have $x = a\infty \oplus b$ for some $a, b \in X$, whereupon

$$\hat{h}(\hat{g}(x)) = \hat{h}(\hat{g}(\infty a \oplus b)) = \hat{h}(\infty g(a) \oplus g(b)) = \infty h(g(a)) \oplus h(g(b)) = \infty a \oplus b = x.$$

By the same argument we have $\hat{g}(\hat{h}(y)) = y$ for all $y \in Y'$, so that $\hat{h}$ is an inverse for $\hat{g}$. Thus, $\hat{g}$ is an isomorphism from X to Y. □

Combining Theorem 5.4 with Theorem 5.1, we obtain an additional description of the $\mathscr{D}$ relation for full matrix semigroups over $\mathbb{T}$.

Theorem 5.5 (Green's $\mathscr{D}$ Relation for $\mathbb{T}^{n\times n}$)**.** *Let $A, B \in \mathbb{T}^{n\times n}$. Then the following are equivalent:*

(i) *$A\mathscr{D}B$ in $\mathbb{T}^{n\times n}$;*
(ii) *$A\mathscr{D}B$ in $\overline{\mathbb{T}}^{n\times n}$ (and the other four equivalent conditions given by Theorem 5.1 in the case $S = \overline{\mathbb{T}}$);*
(iii) *$C_{\mathbb{T}}(A)$ and $C_{\mathbb{T}}(B)$ are isomorphic;*
(iv) *$R_{\mathbb{T}}(A)$ and $R_{\mathbb{T}}(B)$ are isomorphic;*

Proof. The equivalence of (i) and (ii) is part of Theorem 3.6. By Theorem 5.1, (ii) is equivalent to the statement that $C_{\overline{\mathbb{T}}}(A)$ and $C_{\overline{\mathbb{T}}}(B)$ are isomorphic. But $C_{\overline{\mathbb{T}}}(A)$ [respectively, $C_{\overline{\mathbb{T}}}(B)$] is generated as a semimodule over $\overline{\mathbb{T}}$ by the columns of A [B], and hence by $C_{\mathbb{T}}(A)$ [$C_{\mathbb{T}}(B)$]. Hence, by Theorem 5.4, (ii) is equivalent to (iii). The equivalence of (ii) and (iv) is established by a dual argument. □

6. Remarks

We remark briefly on the extent to which our algebraic results, and in particular Theorem 5.1, might apply in wider contexts. Considering Theorem 5.1, we note that while the equivalence of (i), (iv) and (v) is closely bound up with matrix duality, conditions (i), (ii) and (iii) can be shown directly to be equivalent without explicit recourse to duality, by using Theorem 4.2 and Corollary 4.3. These results depend essentially only upon Lemma 4.1. While we proved this lemma using matrix duality, it is likely that an appropriate analogues hold in other semirings for different reasons. The conditions (i), (ii) and (iii) are equivalent, and hence yield characterisations of $\mathscr{D}$ in terms of the isomorphisms of row spaces and isomorphisms of column spaces, for matrices over any such semiring. More generally, we believe that semirings satisfying the condition given in the tropical case by Lemma 4.1 are likely to form a "well-behaved" class, encompassing many examples of interest. As such, they may be deserving of axiomatic study.

Since our methods do not essentially depend upon the matrices considered being square, similar methods should yield corresponding results for Green's relations in the small categories of all finite dimensional matrices (not necessarily square or of uniform size) over $\mathbb{FT}$, $\mathbb{T}$ and $\overline{\mathbb{T}}$ respectively.

Finally, we note that the $\mathscr{J}$ relation and the $\leq_{\mathscr{J}}$ pre-order for tropical matrix semigroups remain poorly understood, and are deserving of further study.

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